

ON THE STABILITY AND CONVERGENCE OF FRACTIONAL-ORDER DIFFERENTIAL EQUATIONS WITH VARIABLE DELAY. APPLIED ANALYSIS AND DIFFERENTIAL EQUATIONS.

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Abstract

This paper investigates the stability and convergence of a class of fractional-order differential equations with variable delay. The fractional derivative is taken in the Caputo sense with order $\alpha \in (0, 1)$, and the delay $\tau(t)$ is a bounded, continuously differentiable function of time. First, the existence and uniqueness of solutions are established using the Banach fixed-point theorem together with a generalized Grönwall-type inequality. Next, sufficient conditions for Mittag-Leffler stability and asymptotic stability of the zero solution are derived by means of a fractional Lyapunov–Razumikhin approach, and the admissible bounds on the delay are stated explicitly. Then, a numerical scheme based on the L1 discretization with linear interpolation of the delayed term is proposed, and its convergence and stability are analyzed. Numerical experiments are presented to illustrate the theoretical results and to show the effect of the fractional order and the delay function on the dynamics of the solutions. The results extend several existing criteria for constant-delay and integer-order systems to the variable-delay fractional setting.

Keywords: Fractional differential equations; Caputo derivative; variable delay; Mittag-Leffler stability; Lyapunov–Razumikhin method; numerical convergence.

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1. Introduction

Fractional calculus, the study of derivatives and integrals of arbitrary order, has become a standard tool for modelling systems with memory and hereditary properties, including viscoelastic materials, anomalous diffusion, control systems and neural networks [1, 2, 3]. Throughout this paper, the Caputo fractional derivative of order $\alpha \in (0, 1)$ of a function x is

$${}^C D^\alpha x(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-s)^{-\alpha} x'(s) ds, \quad t > 0, \quad (1)$$

where $\Gamma(\cdot)$ denotes the Euler gamma function. In many real processes the present rate of change also depends on the past state through a time lag that is not constant, for example due to transmission or processing delays [4]. This leads to the fractional delay differential equation

$$\begin{cases} {}^C D^\alpha x(t) = -Ax(t) + f(t, x(t), x(t - \tau(t))), & t > 0, \\ x(t) = \varphi(t), & t \in [-\tau_{\max}, 0], \end{cases} \quad (2)$$

where $x(t) \in \mathbb{R}^n$, $A \in \mathbb{R}^{n \times n}$ is a positive definite matrix, $\varphi \in C([-\tau_{\max}, 0]; \mathbb{R}^n)$ is the initial function, and $\tau(t)$ is the variable delay. We assume the following conditions.

(H1) The delay satisfies $0 \leq \tau(t) \leq \tau_{\max}$ and $\tau'(t) \leq \rho < 1$ for all $t \geq 0$.

(H2) $f(t, 0, 0) = 0$ and there exist constants $L_1, L_2 \geq 0$ such that

$$\|f(t, u, v) - f(t, \tilde{u}, \tilde{v})\| \leq L_1 \|u - \tilde{u}\| + L_2 \|v - \tilde{v}\| \quad (3)$$

for all $t \geq 0$ and $u, v, \tilde{u}, \tilde{v} \in \mathbb{R}^n$.

Under these conditions, problem (2) is equivalent to the Volterra integral equation

$$x(t) = \varphi(0) + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \left[-Ax(s) + f(s, x(s), x(s - \tau(s))) \right] ds. \quad (4)$$

Its solution behaviour is governed by the Mittag-Leffler function

$$E_\alpha(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + 1)}, \quad z \in \mathbb{C}, \quad (5)$$

which replaces the exponential function of the integer-order case. The zero solution of (2) is called *Mittag-Leffler stable* if there exist constants $\lambda > 0$, $b > 0$ and a locally Lipschitz function $m(\cdot) \geq 0$ with $m(0) = 0$ such that

$$\|x(t)\| \leq \left\{ m(\|\varphi\|_\infty) E_\alpha(-\lambda t^\alpha) \right\}^b, \quad t \geq 0. \quad (6)$$

Since $E_\alpha(-\lambda t^\alpha)$ decays only polynomially, of order $t^{-\alpha}$, as $t \rightarrow \infty$, Mittag-Leffler stability is the natural counterpart of exponential stability for fractional systems [6, 7].

Two difficulties make (2) hard to analyse. First, the Caputo derivative is nonlocal, so classical Lyapunov arguments cannot be applied directly, and one relies on inequalities such as

$${}^C D^\alpha (x^\top(t) P x(t)) \leq 2x^\top(t) P {}^C D^\alpha x(t), \quad P = P^\top > 0, \quad (7)$$

[9]. Second, the variable delay $\tau(t)$ makes the characteristic-equation methods used for constant

delays [8] inapplicable. On the numerical side, on a uniform grid $t_j = jh$ the L1 approximation [10]

$${}^C D^\alpha x(t_n) \approx \frac{h^{-\alpha}}{\Gamma(2-\alpha)} \sum_{k=0}^{n-1} b_k (x_{n-k} - x_{n-k-1}), \quad b_k = (k+1)^{1-\alpha} - k^{1-\alpha}, \quad (8)$$

has a truncation error of order $\mathcal{O}(h^{2-\alpha})$ for smooth solutions, while the delayed term $x(t_n - \tau(t_n))$ falls between grid points and must be handled by interpolation without spoiling this order.

Motivated by these observations, the **key objectives** of this paper are the following.

1. To establish the existence and uniqueness of the solution of (2) under (H1)–(H2), using the Banach fixed-point theorem applied to (4) and a generalized Grönwall inequality.
2. To derive sufficient conditions, expressed through A , L_1 , L_2 , ρ and $\tau_{\max}$, for Mittag-Leffler and asymptotic stability of the zero solution by a fractional Lyapunov–Razumikhin method.
3. To construct an L1-type numerical scheme with interpolation of the delayed term, and to prove its stability and convergence with a rigorous error estimate.
4. To validate the theoretical results by numerical experiments and to examine the effect of the fractional order α and of the delay function $\tau(t)$ on the dynamics.

Table 1 compares the present work with the closely related studies cited above.

Table 1 Comparison of the present work with existing studies

Study	Fractional order	Delay type	Nonlinear terms	Stability analysis	Numerical scheme
Deng et al. [8]	✓	Constant (multiple)	×	✓	×
Li et al. [6]	✓	None	✓	✓	×
Aguila-Camacho et al. [9]	✓	None	✓	✓	×
Lin and Xu [10]	✓	None	×	×	✓
This paper	✓	Variable	✓	✓	✓

The table shows that the cited studies each treat only part of the problem: stability results either ignore delay or assume it constant, and the numerical work does not cover delays. The present paper combines variable delay, nonlinearity, stability analysis and a convergent numerical scheme in one framework.

1.1 Preliminaries and Auxiliary Results

This subsection collects the definitions and lemmas used in the subsequent analysis. Let $\|\cdot\|$ denote the Euclidean norm on $\mathbb{R}^n$, $\|\varphi\|_\infty = \sup_{\theta \in [-\tau_{\max}, 0]} \|\varphi(\theta)\|$, and $\lambda_{\min}(P)$, $\lambda_{\max}(P)$ the extreme eigenvalues of a symmetric matrix P .

[Riemann–Liouville integral] For $\alpha > 0$, the fractional integral of a function $x \in L^1(0, T)$ is

$$I^\alpha x(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} x(s) ds, \quad t \in (0, T]. \quad (9)$$

It satisfies the semigroup property

$$I^\alpha I^\beta x(t) = I^{\alpha+\beta} x(t), \quad \alpha, \beta > 0. \quad (10)$$

[Caputo derivative] For $\alpha \in (0, 1)$ and $x \in AC[0, T]$, the Caputo derivative can be written through (9) as

$${}^C D^\alpha x(t) = I^{1-\alpha} x'(t), \quad (11)$$

and it is the left inverse of the fractional integral in the sense

$$I^\alpha {}^C D^\alpha x(t) = x(t) - x(0), \quad {}^C D^\alpha I^\alpha x(t) = x(t). \quad (12)$$

[Two-parameter Mittag-Leffler function] For $\alpha > 0$ and $\beta > 0$,

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \beta)}, \quad E_{\alpha,1}(z) = E_\alpha(z), \quad (13)$$

with the special cases $E_{1,1}(z) = e^z$ and $E_{2,1}(z) = \cosh \sqrt{z}$. Its Laplace transform is

$$\mathcal{L}\left\{t^{\beta-1} E_{\alpha,\beta}(-\lambda t^\alpha)\right\}(s) = \frac{s^{\alpha-\beta}}{s^\alpha + \lambda}, \quad \operatorname{Re}(s) > \lambda^{1/\alpha}. \quad (14)$$

In particular, the scalar test problem ${}^C D^\alpha x(t) = -\lambda x(t)$, $x(0) = x_0$, has the solution

$$x(t) = x_0 E_\alpha(-\lambda t^\alpha). \quad (15)$$

[Decay of the Mittag-Leffler function] Let $0 < \alpha < 1$ and $\lambda > 0$. Then there exists a constant $C_\alpha > 0$ such that

$$0 < E_\alpha(-\lambda t^\alpha) \leq \frac{C_\alpha}{1 + \lambda t^\alpha}, \quad t \geq 0, \quad (16)$$

and $E_\alpha(-\lambda t^\alpha)$ is completely monotone, that is,

$$(-1)^m \frac{d^m}{dt^m} E_\alpha(-\lambda t^\alpha) \geq 0, \quad m = 0, 1, 2, \dots \quad (17)$$

[Generalized Grönwall inequality] Let $u \geq 0$ be continuous on $[0, T]$, let $a(t)$ be non-decreasing,

and let $b > 0$. If

$$u(t) \leq a(t) + b \int_0^t (t-s)^{\alpha-1} u(s) ds, \quad t \in [0, T], \quad (18)$$

then

$$u(t) \leq a(t) E_\alpha(b \Gamma(\alpha) t^\alpha), \quad t \in [0, T]. \quad (19)$$

[Quadratic Lyapunov inequality] Let $x : [0, \infty) \rightarrow \mathbb{R}^n$ be continuously differentiable and $P = P^\top > 0$. Then for all $\alpha \in (0, 1)$ and $t \geq 0$,

$${}^C D^\alpha (x^\top(t) P x(t)) \leq 2 x^\top(t) P {}^C D^\alpha x(t), \quad (20)$$

and the associated function $V(x) = x^\top P x$ satisfies the norm bounds

$$\lambda_{\min}(P) \|x\|^2 \leq V(x) \leq \lambda_{\max}(P) \|x\|^2. \quad (21)$$

[Mittag-Leffler stability criterion] Let $x = 0$ be an equilibrium of (2). Suppose there exist a continuously differentiable function $V(t, x)$ and constants $\alpha_1, \alpha_2, \alpha_3, a, b > 0$ such that

$$\alpha_1 \|x\|^a \leq V(t, x) \leq \alpha_2 \|x\|^{ab}, \quad (22)$$

$${}^C D^\beta V(t, x(t)) \leq -\alpha_3 \|x(t)\|^{ab}, \quad \beta \in (0, 1). \quad (23)$$

Then the zero solution is Mittag-Leffler stable with respect to (6).

[Fractional Razumikhin condition] Let $V(x) = x^\top P x$ and suppose that, for some $q > 1$,

$$V(x(t+\theta)) < q V(x(t)), \quad \theta \in [-\tau_{\max}, 0], \quad (24)$$

implies

$${}^C D^\alpha V(x(t)) \leq -\gamma V(x(t)) \quad (25)$$

for some $\gamma > 0$. Then $V(x(t)) \leq \|V(\varphi)\|_\infty E_\alpha(-\gamma t^\alpha)$ for all $t \geq 0$, and hence the zero solution of (2) is asymptotically stable.

[Contraction estimate] Define the operator $\mathcal{T}$ on $C([-\tau_{\max}, T]; \mathbb{R}^n)$ by $(\mathcal{T}x)(t) = \varphi(t)$ for $t \in [-\tau_{\max}, 0]$ and, for $t > 0$, by the right-hand side of (4). With the weighted norm

$$\|x\|_\theta = \sup_{t \in [-\tau_{\max}, T]} e^{-\theta t} \|x(t)\|, \quad \theta > 0, \quad (26)$$

and using

$$\int_0^t (t-s)^{\alpha-1} e^{\theta s} ds \leq \Gamma(\alpha) \theta^{-\alpha} e^{\theta t}, \quad (27)$$

we obtain, under (H1)–(H2),

$$\|\mathcal{T}x - \mathcal{T}y\|_\theta \leq \frac{\|A\| + L_1 + L_2}{\theta^\alpha} \|x - y\|_\theta. \quad (28)$$

Hence $\mathcal{T}$ is a contraction whenever

$$\theta > (\|A\| + L_1 + L_2)^{1/\alpha}. \quad (29)$$

[Properties of the L1 scheme] Let $b_k = (k+1)^{1-\alpha} - k^{1-\alpha}$ for $k \geq 0$. Then

$$1 = b_0 > b_1 > b_2 > \cdots > 0, \quad \lim_{k \rightarrow \infty} b_k = 0, \quad \sum_{k=0}^{n-1} b_k = n^{1-\alpha}. \quad (30)$$

Moreover, for $x \in C^2[0, T]$, the truncation error

$$R^n = \left| {}^C D^\alpha x(t_n) - \frac{h^{-\alpha}}{\Gamma(2-\alpha)} \sum_{k=0}^{n-1} b_k (x_{n-k} - x_{n-k-1}) \right| \leq C_\alpha \max_{t \in [0, T]} |x''(t)| h^{2-\alpha}. \quad (31)$$

[Interpolation of the delayed term] Let $t_n - \tau(t_n) \in [t_m, t_{m+1}]$ and define the linear interpolant

$$\Pi x(t_n - \tau_n) = (1 - \mu_n) x_m + \mu_n x_{m+1}, \quad \mu_n = \frac{t_n - \tau_n - t_m}{h} \in [0, 1], \quad (32)$$

where $\tau_n = \tau(t_n)$. Then, for $x \in C^2[-\tau_{\max}, T]$,

$$|x(t_n - \tau_n) - \Pi x(t_n - \tau_n)| \leq \frac{h^2}{8} \max_t |x''(t)|. \quad (33)$$

Since $h^2 \leq h^{2-\alpha}$ for $h \leq 1$, the interpolation error does not reduce the accuracy $\mathcal{O}(h^{2-\alpha})$ of the L1 scheme.

Combining (19) and (28), the solution of (2) satisfies the a priori bound

$$\|x(t)\| \leq \|\varphi\|_\infty E_\alpha \left((\|A\| + L_1 + L_2) \Gamma(\alpha) t^\alpha \right), \quad t \in [0, T], \quad (34)$$

which shows that solutions cannot blow up in finite time.

2. Methodology

The study is analytical and computational. It has four stages: (i) well-posedness of (2), (ii) stability of the zero solution by a Lyapunov–Razumikhin argument, (iii) construction and analysis of a numerical scheme, and (iv) numerical validation. Throughout, A is symmetric positive definite, $\lambda := \lambda_{\min}(A)$, and (H1)–(H2) hold.

2.1 Well-Posedness by Picard Iteration

Let $\mathcal{T}$ be the operator of Lemma 1.1. Starting from $x^{(0)}(t) = \varphi(0)$ for $t \geq 0$ and $x^{(0)} = \varphi$ on $[-\tau_{\max}, 0]$, define the Picard sequence

$$x^{(k+1)} = \mathcal{T}x^{(k)}, \quad k = 0, 1, 2, \dots \quad (35)$$

By (28), $\mathcal{T}$ is a contraction in $\|\cdot\|_{\theta}$ with constant

$$q_{\theta} = \frac{\|A\| + L_1 + L_2}{\theta^{\alpha}} < 1, \quad (36)$$

so (35) converges to the unique fixed point x^* of (4), with the a priori estimate

$$\|x^{(k)} - x^*\|_{\theta} \leq \frac{q_{\theta}^k}{1 - q_{\theta}} \|x^{(1)} - x^{(0)}\|_{\theta}. \quad (37)$$

Since θ is arbitrary subject to (29), the solution is unique on every interval $[0, T]$.

2.2 Stability Analysis by the Lyapunov–Razumikhin Method

Take $V(x) = x^{\top}x = \|x\|^2$. By Lemma 1.1 and (2),

$${}^CD^{\alpha}V(x(t)) \leq 2x^{\top}(t) \left[-Ax(t) + f(t, x(t), x(t - \tau(t))) \right]. \quad (38)$$

Using $x^{\top}Ax \geq \lambda\|x\|^2$, $f(t, 0, 0) = 0$ and (3),

$${}^CD^{\alpha}V(x(t)) \leq -2\lambda\|x(t)\|^2 + 2L_1\|x(t)\|^2 + 2L_2\|x(t)\| \|x(t - \tau(t))\|. \quad (39)$$

Now assume the Razumikhin condition (24) for some $q > 1$, that is,

$$\|x(t + \theta)\|^2 < q\|x(t)\|^2, \quad \theta \in [-\tau_{\max}, 0]. \quad (40)$$

Then $\|x(t - \tau(t))\| < \sqrt{q}\|x(t)\|$, and (39) becomes

$${}^CD^{\alpha}V(x(t)) \leq -\gamma V(x(t)), \quad \gamma := 2(\lambda - L_1 - \sqrt{q}L_2). \quad (41)$$

[Mittag-Leffler stability] If

$$\lambda_{\min}(A) > L_1 + L_2, \quad (42)$$

then there is $q > 1$ with $\gamma > 0$ in (41), and the zero solution of (2) satisfies

$$\|x(t)\| \leq \|\varphi\|_{\infty} \left[E_{\alpha}(-\gamma t^{\alpha}) \right]^{1/2}, \quad t \geq 0, \quad (43)$$

which is (6) with $m(s) = s^2$ and $b = \frac{1}{2}$. Together with (16), $\|x(t)\| = \mathcal{O}(t^{-\alpha/2})$ as $t \rightarrow \infty$.

Proof. Because (42) is a strict inequality, choose $q > 1$ close enough to 1 that $\sqrt{q} L_2 < \lambda - L_1$, so $\gamma > 0$. Then (41) holds whenever (40) does, and Lemma 1.1 gives $V(x(t)) \leq \|\varphi\|_\infty^2 E_\alpha(-\gamma t^\alpha)$. Taking square roots gives (43). $\square$

2.3 Numerical Scheme

Let $h = T/N$, $t_j = jh$, $x_j \approx x(t_j)$ and $\tau_n = \tau(t_n)$. Set

$$c_\alpha = \frac{h^{-\alpha}}{\Gamma(2-\alpha)}, \quad b_k = (k+1)^{1-\alpha} - k^{1-\alpha}, \quad d_k = b_{k-1} - b_k > 0 \quad (k \geq 1). \quad (44)$$

Summation by parts gives the equivalent form of the L1 approximation (8):

$$\sum_{k=0}^{n-1} b_k (x_{n-k} - x_{n-k-1}) = x_n - \sum_{k=1}^{n-1} d_k x_{n-k} - b_{n-1} x_0, \quad (45)$$

where $\sum_{k=1}^{n-1} d_k = 1 - b_{n-1}$. The delayed value is approximated as follows. Let $s_n = t_n - \tau_n$. If $s_n \leq 0$, use $\tilde{x}_n = \varphi(s_n)$. Otherwise set

$$m = \min \left\{ \left\lfloor \frac{s_n}{h} \right\rfloor, n-1 \right\}, \quad \mu_n = \frac{s_n}{h} - m \in [0, 1], \quad \tilde{x}_n = (1 - \mu_n)x_m + \mu_n x_{m+1}. \quad (46)$$

Replacing ${}^C D^\alpha x(t_n)$ and $x(t_n - \tau_n)$ in (2) gives the fully discrete scheme

$$(c_\alpha I + A)x_n = c_\alpha \mathcal{H}_n + f(t_n, x_n, \tilde{x}_n), \quad n = 1, \dots, N, \quad (47)$$

with the memory term

$$\mathcal{H}_n = \sum_{k=1}^{n-1} d_k x_{n-k} + b_{n-1} x_0, \quad x_0 = \varphi(0). \quad (48)$$

Equation (47) is implicit in x_n : through f , and through $\tilde{x}_n$ when $m+1 = n$. It is solved by the fixed-point iteration

$$y^{(j+1)} = (c_\alpha I + A)^{-1} \left[c_\alpha \mathcal{H}_n + f(t_n, y^{(j)}, \Psi_n(y^{(j)})) \right], \quad y^{(0)} = x_{n-1}, \quad (49)$$

where $\Psi_n(y)$ is $\tilde{x}_n$ with x_n replaced by y . The map $y \mapsto f(t_n, y, \Psi_n(y))$ is Lipschitz with constant at most $L_1 + L_2 \mu_n \leq L_1 + L_2$, and $\|(c_\alpha I + A)^{-1}\| \leq (c_\alpha + \lambda)^{-1}$. Hence (49) contracts with factor

$$q_{\text{fp}} = \frac{L_1 + L_2}{c_\alpha + \lambda} < 1 \quad (50)$$

whenever $h < [\Gamma(2 - \alpha)(L_1 + L_2 - \lambda)]^{-1/\alpha}$, and for every h if $\lambda > L_1 + L_2$. The iteration error obeys

$$\|y^{(j)} - x_n\| \leq \frac{q_{\text{fp}}^j}{1 - q_{\text{fp}}} \|y^{(1)} - y^{(0)}\|. \quad (51)$$

2.4 Algorithm

Algorithm 1 L1–interpolation scheme with fixed-point iteration for (2)

Require: $\alpha \in (0, 1)$, A , f , $\tau(\cdot)$, $\varphi(\cdot)$, T , N , tolerance ε , iteration cap $J_{\max}$

Ensure: Approximations $\{x_n\}_{n=0}^N \approx \{x(t_n)\}_{n=0}^N$

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0:  $h \leftarrow T/N$ ;  $t_j \leftarrow jh$ ;  $c_\alpha \leftarrow h^{-\alpha}/\Gamma(2 - \alpha)$ 
0:  $b_k \leftarrow (k + 1)^{1-\alpha} - k^{1-\alpha}$  for  $k = 0, \dots, N - 1$ ;  $d_k \leftarrow b_{k-1} - b_k$  for  $k = 1, \dots, N - 1$ 
0: Factorize  $W \leftarrow c_\alpha I + A$  once;  $x_0 \leftarrow \varphi(0)$ 
0: for  $n = 1, 2, \dots, N$  do
0:    $\mathcal{H}_n \leftarrow \sum_{k=1}^{n-1} d_k x_{n-k} + b_{n-1} x_0$ 
0:    $s_n \leftarrow t_n - \tau(t_n)$ 
0:   if  $s_n > 0$  then
0:      $m \leftarrow \min\{\lfloor s_n/h \rfloor, n - 1\}$ ;  $\mu_n \leftarrow s_n/h - m$ 
0:   end if
0:    $y^{(0)} \leftarrow x_{n-1}$ ;  $j \leftarrow 0$ 
0:   repeat
0:     if  $s_n \leq 0$  then
0:        $\tilde{x} \leftarrow \varphi(s_n)$ 
0:     else
0:        $\tilde{x} \leftarrow (1 - \mu_n)x_m + \mu_n x_{m+1}$ , using  $x_n := y^{(j)}$  if  $m + 1 = n$ 
0:     end if
0:     Solve  $W y^{(j+1)} = c_\alpha \mathcal{H}_n + f(t_n, y^{(j)}, \tilde{x})$ 
0:      $j \leftarrow j + 1$ 
0:   until  $\|y^{(j)} - y^{(j-1)}\| \leq \varepsilon$  or  $j = J_{\max}$ 
0:    $x_n \leftarrow y^{(j)}$ 
0: end for
0: return  $\{x_n\}_{n=0}^N$ 

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Line 5 costs $\mathcal{O}(n)$ per step, so the total number of memory-sum operations is

$$\sum_{n=1}^N (n - 1) = \frac{N(N - 1)}{2} = \mathcal{O}(N^2), \quad (52)$$

plus $\mathcal{O}(N J)$ evaluations of f , where J is the average number of inner iterations. Storage is $\mathcal{O}(N)$.

2.5 Stability and Convergence of the Scheme

[Discrete stability] Under (42), scheme (47) has a unique solution for every $h > 0$, and

$$\|x_n\| \leq \|\varphi\|_\infty, \quad n = 0, 1, \dots, N. \quad (53)$$

Proof. Existence and uniqueness follow from (50) with $q_{fp} < 1$. Let $M = \max\{\|\varphi\|_\infty, \max_j \|x_j\|\}$. If $M = \|\varphi\|_\infty$ we are done. Otherwise $M = \|x_{n^*}\|$ for some $n^* \geq 1$. Taking norms in (47) at n^* and using $\|\tilde{x}_{n^*}\| \leq M$,

$$(c_\alpha + \lambda)M \leq c_\alpha(1 - b_{n^*-1})M + c_\alpha b_{n^*-1}\|x_0\| + (L_1 + L_2)M. \quad (54)$$

Rearranging,

$$(c_\alpha b_{n^*-1} + \lambda - L_1 - L_2)M \leq c_\alpha b_{n^*-1}\|x_0\|. \quad (55)$$

Since $\lambda - L_1 - L_2 > 0$, this gives $M \leq \|x_0\| \leq \|\varphi\|_\infty$, a contradiction. Hence $M = \|\varphi\|_\infty$. $\square$

[Convergence] Let $x \in C^2[0, T]$ with $M_2 = \max_{[0, T]} \|x''(t)\|$, let $h \leq 1$, and let

$$\kappa := \lambda - L_1 - L_2 + \frac{T^{-\alpha}}{\Gamma(1 - \alpha)} > 0. \quad (56)$$

Then

$$\max_{0 \leq n \leq N} \|x(t_n) - x_n\| \leq \frac{C_\alpha M_2 + L_2 M_2 / 8}{\kappa} h^{2-\alpha}. \quad (57)$$

Proof. Let $e_n = x(t_n) - x_n$, so $e_n = 0$ for $n \leq 0$. Subtracting (47) from the equation satisfied by the exact solution at t_n , and using Lemmas 1.1 and 1.1, gives the error equation

$$c_\alpha \left[e_n - \sum_{k=1}^{n-1} d_k e_{n-k} \right] + A e_n = F_n + \tilde{R}_n, \quad (58)$$

with

$$\|F_n\| \leq L_1 \|e_n\| + L_2 \max\{\|e_m\|, \|e_{m+1}\|\}, \quad \|\tilde{R}_n\| \leq (C_\alpha M_2 + \tfrac{1}{8} L_2 M_2) h^{2-\alpha} =: C_R h^{2-\alpha}, \quad (59)$$

where $h^2 \leq h^{2-\alpha}$ was used. Let $\epsilon = \max_n \|e_n\| = \|e_{n^*}\|$. Taking norms at n^* and using $\sum_{k=1}^{n^*-1} d_k = 1 - b_{n^*-1}$,

$$(c_\alpha b_{n^*-1} + \lambda - L_1 - L_2) \epsilon \leq C_R h^{2-\alpha}. \quad (60)$$

By the mean value theorem and monotonicity of b_k ,

$$b_{n^*-1} \geq b_{N-1} \geq (1 - \alpha) N^{-\alpha}, \quad (61)$$

so, using $(1 - \alpha)/\Gamma(2 - \alpha) = 1/\Gamma(1 - \alpha)$,

$$c_\alpha b_{n^*-1} \geq \frac{h^{-\alpha}(1 - \alpha)}{\Gamma(2 - \alpha)} \left(\frac{T}{h}\right)^{-\alpha} = \frac{T^{-\alpha}}{\Gamma(1 - \alpha)}. \quad (62)$$

Substituting (62) in (60) gives $\kappa \epsilon \leq C_R h^{2-\alpha}$, which is (57). $\square$

On a uniform mesh the order $2 - \alpha$ requires $x \in C^2[0, T]$. For general fractional problems $x''(t) \sim t^{\alpha-2}$ near $t = 0$, and a graded mesh $t_j = T(j/N)^r$ with $r \geq (2 - \alpha)/\alpha$ restores the order.

2.6 Numerical Validation Design

To verify Theorem 2.5, take the scalar problem ($n = 1$) with a manufactured solution $x(t) = t^2 + 1$ and data

$$A = a = 2, \quad f(t, u, v) = \beta \sin v + g(t), \quad \beta = 0.5, \quad \tau(t) = 0.5 + 0.2 \sin t, \quad (63)$$

so $L_1 = 0$, $L_2 = 0.5$, $\tau_{\max} = 0.7$ and $\rho = 0.2$. The forcing term

$$g(t) = \frac{2t^{2-\alpha}}{\Gamma(3-\alpha)} + a(t^2 + 1) - \beta \sin((t - \tau(t))^2 + 1) \quad (64)$$

makes $x(t) = t^2 + 1$ the exact solution with $\varphi(t) = t^2 + 1$ on $[-\tau_{\max}, 0]$, and (42) holds since $2 > 0.5$. Accuracy is measured by

$$E(h) = \max_{0 \leq n \leq N} |x(t_n) - x_n|, \quad \text{EOC} = \frac{\ln(E(h)/E(h/2))}{\ln 2}, \quad (65)$$

and Theorem 2.5 predicts $\text{EOC} \rightarrow 2 - \alpha$. Stability is checked with the unforced problem ($g \equiv 0$) by comparing the computed decay of $|x_n|$ with the predicted envelope

$$|x(t_n)| \leq \|\varphi\|_{\infty} \left[E_{\alpha}(-\gamma t_n^{\alpha}) \right]^{1/2}, \quad \gamma = 2(\lambda - L_1 - \sqrt{q} L_2), \quad (66)$$

for $\alpha \in \{0.3, 0.5, 0.7, 0.9\}$ and several delay functions $\tau(t)$.

3. Results and Discussion

All experiments use the scalar test problem (63) with $a = 2$, $\beta = 0.5$ and $\tau(t) = 0.5 + 0.2 \sin t$, so that $\lambda = 2 > L_1 + L_2 = 0.5$ and condition (42) holds. The scheme (47) was implemented as in Algorithm 1 with inner tolerance $\varepsilon = 10^{-14}$. Convergence is studied on $[0, 1]$ with the manufactured solution $x(t) = t^2 + 1$ and $N \in \{20, 40, 80, 160, 320, 640\}$. Stability is studied on $[0, 10]$ with $N = 1000$ for the unforced problem ($g \equiv 0$, $\varphi \equiv 1$). Mittag-Leffler values were computed from the series (5) in high-precision arithmetic.

3.1 Convergence of the Numerical Scheme

Table 2 reports the maximum error $E(h)$ and the experimental order of convergence (EOC) defined in (65) for $\alpha \in \{0.3, 0.5, 0.7, 0.9\}$.

Table 2 Maximum error $E(h)$ and EOC for the test problem (63) on $[0, 1]$, $h = 1/N$

N	$\alpha = 0.3$		$\alpha = 0.5$		$\alpha = 0.7$		$\alpha = 0.9$	
	$E(h)$	EOC	$E(h)$	EOC	$E(h)$	EOC	$E(h)$	EOC
20	6.51-4	–	1.97-3	–	5.37-3	–	1.38-2	–
40	2.06-4	1.66	7.07-4	1.48	2.20-3	1.29	6.49-3	1.09
80	6.49-5	1.67	2.52-4	1.49	8.97-4	1.29	3.04-3	1.10
160	2.03-5	1.68	8.97-5	1.49	3.65-4	1.30	1.42-3	1.10
320	6.33-6	1.68	3.19-5	1.49	1.49-4	1.30	6.62-4	1.10
640	1.97-6	1.68	1.13-5	1.50	6.04-5	1.30	3.09-4	1.10
Theory 2 – α	1.70		1.50		1.30		1.10	

[width=0.8height=7cm,xlabel= h ,ylabel= $E(h)$, legend pos=south east,grid=both,legend style=font=] [mark=o] coordinates (0.05,6.509e-4)(0.025,2.062e-4)(0.0125,6.488e-5)(0.00625,2.028e-5)(0.003125,6.328e-6)(0.0015625,1.969e-6); [mark=square] coordinates (0.05,1.970e-3)(0.025,7.073e-4)(0.0125,2.523e-4)(0.00625,8.971e-5)(0.003125,3.188e-5)(0.0015625,1.131e-5); [mark=triangle] coordinates (0.05,5.366e-3)(0.025,2.198e-3)(0.0125,8.966e-4)(0.00625,3.651e-4)(0.003125,1.485e-4)(0.0015625,6.036e-5); [mark=diamond] coordinates (0.05,1.381e-2)(0.025,6.487e-3)(0.0125,3.036e-3)(0.00625,1.419e-3)(0.003125,6.624e-4)(0.0015625,3.091e-4); $\alpha = 0.3, \alpha = 0.5, \alpha = 0.7, \alpha = 0.9$

Figure 1: Error $E(h)$ versus step size h (log–log scale).

For every α the EOC increases monotonically with refinement and approaches the theoretical value $2 - \alpha$ of Theorem 2.5 from below (for example 1.50 at $\alpha = 0.5$ and 1.30 at $\alpha = 0.7$ on the finest mesh). The error grows with α for a given h , because the L1 constant C_α and the memory term become larger as $\alpha \rightarrow 1$. The variable delay, which forces the interpolation (46), does not reduce the order, which agrees with Lemma 1.1.

3.2 Stability and Decay of the Solution

Table 3 compares the computed $|x(t)|$ for the unforced problem with the envelope (66) evaluated with $\gamma = 2$, which corresponds to $q = 4$ in (41).

In all cases the solution decreases monotonically to zero and stays below the envelope, which confirms asymptotic stability of the zero solution. The decay is algebraic and not exponential:

Table 3 Computed $|x(t)|$ / envelope $[E_\alpha(-2t^\alpha)]^{1/2}$ for $\varphi \equiv 1$

α	$t = 1$	$t = 2$	$t = 5$	$t = 10$
0.3	0.383 / 0.539	0.319 / 0.498	0.252 / 0.446	0.213 / 0.409
0.5	0.363 / 0.505	0.264 / 0.435	0.166 / 0.351	0.118 / 0.297
0.7	0.341 / 0.462	0.207 / 0.355	0.090 / 0.248	0.051 / 0.190
0.9	0.315 / 0.404	0.144 / 0.239	0.028 / 0.126	0.011 / 0.086

for $\alpha = 0.3$ and 0.5 the quantity $t^\alpha|x(t)|$ stays between 0.37 and 0.43 on $2 \leq t \leq 10$, consistent with the $t^{-\alpha}$ rate implied by (16), while for $\alpha = 0.7$ and 0.9 the solution decays faster than $t^{-\alpha}$ over the computed window.

The envelope is sensitive to the chosen rate. With $q \rightarrow 1$ in (41) the rate is $\gamma \approx 3$, and this envelope was *exceeded* for $\alpha = 0.9$ at 4 of 25 sampled times in $[0, 10]$ (by at most 8.6% , near $t = 1$). No violation occurred for $\alpha = 0.7$ with any $\gamma \in \{3, 2, 1.5, 1\}$, nor for $\alpha = 0.9$ with $\gamma \leq 2$. Hence the rate in Theorem 2.2 must be linked to q and to $\tau_{\max}$ (as in the classical Razumikhin theorem, where $q \geq e^{\gamma\tau_{\max}}$), and the value $\gamma = 2$ used in Table 3 is the conservative one supported by these experiments.

3.3 Effect of the Delay Function

Table 4 shows $|x(t)|$ for $\alpha = 0.5$ and four delay functions with the same initial function and no forcing.

Table 4 Values of $|x(t)|$ for $\alpha = 0.5$ and different delays

Delay $\tau(t)$	$t = 1$	$t = 3$	$t = 5$	$t = 10$
$0.5 + 0.2 \sin t$ (variable)	0.3631	0.2150	0.1656	0.1182
0.7 (constant)	0.3672	0.2170	0.1679	0.1188
0.3 (constant)	0.3379	0.2082	0.1638	0.1173
0.01 (almost no delay)	0.3207	0.2034	0.1614	0.1164

A larger delay slows the early decay (at $t = 1$: 0.321 for $\tau = 0.01$ versus 0.367 for $\tau = 0.7$), but the differences shrink to about 2% by $t = 10$, and the variable delay behaves like a constant delay of similar mean size. This agrees with the delay-independent form of condition (42): the delay affects the transient but not the asymptotic behaviour.

3.4 Discussion

The results support three conclusions. First, the L1–interpolation scheme attains the predicted order $2 - \alpha$ for a smooth solution with a genuinely time-varying delay. Second, condition (42) gives a stable computed solution for all four orders tested. Third, the stability rate must be chosen

conservatively when α is close to 1. The experiments are limited to a scalar equation with one nonlinearity and one parameter set, and the smooth exact solution avoids the initial singularity that appears for general data (see the Remark after Theorem 2.5).

4. Conclusion

This paper studied the stability and convergence of Caputo fractional differential equations with variable delay. Existence and uniqueness of the solution were established through a weighted-norm contraction argument. A Lyapunov–Razumikhin analysis showed that the zero solution is Mittag-Leffler stable when $\lambda_{\min}(A) > L_1 + L_2$. An L1 scheme with linear interpolation of the delayed term was constructed, and it was proved to be stable and convergent of order $\mathcal{O}(h^{2-\alpha})$.

Numerical experiments confirmed the convergence order for $\alpha \in \{0.3, 0.5, 0.7, 0.9\}$, with experimental orders 1.68, 1.50, 1.30 and 1.10 on the finest mesh. They also showed that the computed solutions decay monotonically and algebraically, and that the delay mainly affects the transient phase. The experiments indicated that the decay rate in the stability theorem depends on the Razumikhin constant q and the delay bound, and this dependence should be made explicit.

The main limitations are the scalar test problem, the smoothness assumption $x \in C^2[0, T]$, and the $\mathcal{O}(N^2)$ cost of the memory sum. Future work includes systems of equations, graded meshes for weakly singular solutions, fast methods that reduce the cost of the history term, and Lyapunov–Krasovskii functionals giving explicit delay-dependent bounds.

References

- [1] I. Podlubny, *Fractional Differential Equations*, Academic Press, San Diego, 1999.
- [2] A. A. Kilbas, H. M. Srivastava and J. J. Trujillo, *Theory and Applications of Fractional Differential Equations*, North-Holland Mathematics Studies, vol. 204, Elsevier, Amsterdam, 2006.
- [3] K. Diethelm, *The Analysis of Fractional Differential Equations*, Lecture Notes in Mathematics, vol. 2004, Springer, Berlin, 2010.
- [4] J. K. Hale and S. M. Verduyn Lunel, *Introduction to Functional Differential Equations*, Applied Mathematical Sciences, vol. 99, Springer, New York, 1993.
- [5] V. Lakshmikantham and A. S. Vatsala, Basic theory of fractional differential equations, *Nonlinear Analysis: Theory, Methods & Applications*, **69**(8) (2008), 2677–2682.
- [6] Y. Li, Y. Chen and I. Podlubny, Mittag-Leffler stability of fractional order nonlinear dynamic systems, *Automatica*, **45**(8) (2009), 1965–1969.

- [7] Y. Li, Y. Chen and I. Podlubny, Stability of fractional-order nonlinear dynamic systems: Lyapunov direct method and generalized Mittag-Leffler stability, *Computers & Mathematics with Applications*, **59**(5) (2010), 1810–1821.
- [8] W. Deng, C. Li and J. Lü, Stability analysis of linear fractional differential system with multiple time delays, *Nonlinear Dynamics*, **48**(4) (2007), 409–416.
- [9] N. Aguila-Camacho, M. A. Duarte-Mermoud and J. A. Gallegos, Lyapunov functions for fractional order systems, *Communications in Nonlinear Science and Numerical Simulation*, **19**(9) (2014), 2951–2957.
- [10] Y. Lin and C. Xu, Finite difference/spectral approximations for the time-fractional diffusion equation, *Journal of Computational Physics*, **225**(2) (2007), 1533–1552.
- [11] H. Ye, J. Gao and Y. Ding, A generalized Grönwall inequality and its application to a fractional differential equation, *Journal of Mathematical Analysis and Applications*, **328**(2) (2007), 1075–1081.